

ON REGULARIZED ELLIPTIC GENERA OF ALE SPACES

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ABSTRACT. We define regularized elliptic genera of ALE space of type A by taking some regularized nonequivariant limits of their equivariant elliptic genera with respect to some torus actions. They turn out to be multiples of the elliptic genus of a K3 surface.

1. INTRODUCTION

In an early paper [16] we have studied equivariant elliptic genera of toric Calabi-Yau 3-folds. In this paper we will study the case of toric Calabi-Yau 2-folds. They are ALE spaces of type A. The study of elliptic genera of noncompact Calabi-Yau manifolds was initiated about ten about years by Eguchi and Sugawara [6, 7]. The case of ALE spaces was further studied by Eguchi-Sugawara-Taormina [8]. Predictions for elliptic genera of toric Calabi-Yau 3-folds and ALE spaces were made in these papers. Equivariant elliptic genera of noncompact spaces were studied around the same time in the physics literature [11] and the mathematical literature [12, 13, 14]. Prompted by some recent work on equivariant elliptic genera by physicists [10, 9], we attempt to verify the predictions of Eguchi and Sugawara based on equivariant elliptic genera of toric Calabi-Yau manifolds in and this paper. It turns out to be successful in the case of toric Calabi-Yau 3-folds [16], but not so successful in the case of toric Calabi-Yau surfaces studied in this paper. Nevertheless we believe this study has some independent interest which might be useful for other problems.

A toric Calabi-Yau n -fold is a noncompact complex manifold constructed by adding some lower dimensional complex tori to the n -dimensional complex torus $(\mathbb{C}^*)^n$ according to some combinatorial data. It admits natural n -torus action with isolated fixed points. One can define its equivariant elliptic genus by adding up the contributions from the fixed points. The equivariant elliptic genus then depends on n extra variables t_1, \dots, t_n , which are linear coordinates on the Lie algebra of the n -torus. Our idea is to get rid of these extra variables in some suitable way. First of all, we restrict to an $n - 1$ -dimensional subtorus which fixes a holomorphic volume form. This not only reduces the

number of extra variables by one, but also simplifies the expressions to work with.

In the case of toric Calabi-Yau 3-fold, we have used a simple averaging method [16]. Unfortunately, this method no longer works for very simple reasons in the case of toric Calabi-Yau surfaces or four-olds. We will treat in this paper the case of toric Calabi-Yau surfaces by defining a regularized nonequivariant limit. More precisely, we will analyze the singularities of the equivariant elliptic genera of ALE spaces of type A when the extra variable is approaching zero, we subtract the singular terms then take the limit. (In doing so we make use of some results for the equivariant elliptic genus of \mathbb{C}^2 . We present some alternative proofs of them to gain more understandings from different perspectives.) The results turn out to be proportional to the elliptic genus of a K3 surface. So they are not the same as the elliptic genera of ALE spaces corresponding to the decompactified elliptic genus of K3 surfaces studied in [8]. Nevertheless they should also be interesting because they are weak Jacobi forms as in the compact case.

We arrange the rest of the paper as follows. In §2 we recall the 2-torus actions on ALE spaces of type A. In §3 we present the modular transformation properties of their equivariant elliptic genera. In §4 we introduce their regularized elliptic genera after analyzing the singularities of their equivariant elliptic genera.

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2. A TORUS ACTION ON ALE SPACES OF TYPE A

In this section we recall the definition of some torus actions on ALE spaces of type A. See [15, Example 4.1] and the references therein.

2.1. The ALE Space of type A_{n-1} . Such a space can be obtained by gluing n copies of \mathbb{C}^2 as follows. For $i = 0, 1, \dots, n-1$, denote by U_i the i -th copy of \mathbb{C}^2 and let (x_i, y_i) be the linear coordinates on it. Then U_i and U_{i+1} are glued together by the following formula for change of coordinates:

$$(1) \quad x_{i+1} = x_i^2 y_i, \quad y_{i+1} = x_i^{-1},$$

identifying $U_i - \{x_i = 0\}$ with $U_{i+1} - \{y_{i+1} = 0\}$. It is easy to see that

$$(2) \quad dx_{i+1} \wedge dy_{i+1} = -dx_i \wedge dy_i,$$

so one has a holomorphic volume form Ω on $X_n = U_0 \cup \dots \cup U_{n-1}$ such that on each U_i ,

$$(3) \quad \Omega = (-1)^{i-1} dx_i \wedge dy_i.$$

For $i = 1, \dots, n$, let C_i be $\{(x_{i-1}, 0) \in U_{i-1} \mid x_{i-1} \neq 0\}$ glued with $\{(0, y_i) \in U_i \mid y_i \neq 0\}$. Then C_i is a copy of \mathbb{P}^1 . By (1) one sees that the normal bundle of C_i in X_n is -2 , and so the self-intersection number of C_i is -2 . One can also see that

$$(4) \quad C_i \cdot C_j = \begin{cases} -2, & \text{if } i = j, \\ 1, & \text{if } i = j + 1 \text{ or } j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $i = 1, \dots, n$, let $V_i = U_i - \{x_i = 0\} - \{y_i = 0\} \cong (\mathbb{C}^*)^2$. Then the gluing identifies all V_i 's with each other, and X_n is obtained by adding copies of $(\mathbb{C}^*)^1$ and $(\mathbb{C}^*)^0$, i.e., it is an example of toric varieties.

Next we recall the natural map $\pi : X_n \rightarrow \mathbb{C}^2/\mathbb{Z}_n$, where \mathbb{Z}_n acts on \mathbb{C}^2 as follows:

$$(5) \quad \xi_n \cdot (x, y) = (\xi_n x, \xi_n^{-1} y),$$

where $\xi_n = e^{2\pi i/n}$. In U_i , π is given by:

$$(6) \quad x = \frac{x_i y_i}{(x_i^i y_i^{i+1})^{1/n}} = x_i^{1-i/n} y_i^{1-(i+1)/n}, \quad y = (x_i^i y_i^{i+1})^{1/n}.$$

One can easily check that this is well-defined by (1). Furthermore, π maps all the curves C_i to $[(0, 0)] \in \mathbb{C}^2/\mathbb{Z}_n$. On V_i we have an inverse map of π :

$$(7) \quad x_i = \frac{x^{i+1}}{y^{n-i-1}}, \quad y_i = \frac{y^{n-i}}{x^i}.$$

2.2. Torus action on ALE spaces. Consider the following two-torus action on \mathbb{C}^2 :

$$(8) \quad (s_1, s_2) \cdot (x, y) = (s_1 x, s_2 y).$$

It induces the following T^2 -action on X_n by (7):

$$(9) \quad (s_1, s_2) \cdot (x_i, y_i) = (s_1^{-i-1} s_2^{n-i-1} x_i, s_1^i s_2^{-n+i} y_i)$$

on U_i . One can directly check that these actions on U_i glued together to an action on X_n by (1).

2.3. Weight decomposition at the fixed points. It is clear that the T^2 -action has n isolated fixed points p_i , $i = 0, \dots, n-1$, where $p_i \in U_i$ is given by $x_i = y_i = 0$. Since (x_i, y_i) are linear coordinates on U_i , the weight decomposition at p_i is given by (9).

3. EQUIVARIANT ELLIPTIC GENERA OF ALE SPACES

In this section we study the modular transformation properties of equivariant genera of ALE spaces of type A by Jacobi theta function. For reference on equivariant elliptic genera of noncompact complex manifold, the reader can consult [12].

3.1. Equivariant elliptic genera of ALE spaces. Using information on the fixed points and weight decompositions there, the equivariant elliptic genus of X_r is given by the Lefschetz fixed point contributions [1]:

$$(10) \quad \begin{aligned} Z_{X_r}(\tau, z; t_1, t_2) \\ = \sum_{j=0}^{r-1} \frac{\theta_1(\tau, z - (j+1)t_1 + (r-j-1)t_2)}{\theta_1(\tau, -(j+1)t_1 + (r-j-1)t_2)} \\ \cdot \frac{\theta_1(\tau, z + jt_1 + (j-r)t_2)}{\theta_1(\tau, jt_1 + (j-r)t_2)}, \end{aligned}$$

where θ_1 is the theta function defined by:

$$(11) \quad \theta_1(\tau, z) = iq^{1/8}y^{-1/2} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1})(1 - y^{-1}q^m).$$

For example,

$$(12) \quad \begin{aligned} Z_{X_2}(z, \tau; t_1, t_2) &= \frac{\theta_1(z - t_1 + t_2, \tau)}{\theta_1(-t_1 + t_2, \tau)} \frac{\theta_1(z - 2t_2, \tau)}{\theta_1(-2t_2, \tau)} \\ &+ \frac{\theta_1(z - 2t_1, \tau)}{\theta_1(-2t_1, \tau)} \cdot \frac{\theta_1(z + t_1 - t_2, \tau)}{\theta_1(t_1 - t_2, \tau)}. \end{aligned}$$

Such expressions were obtained in [9] in the setting of gauged linear sigma model by the method developed in [2].

3.2. Modular transformation properties of equivariant elliptic genera of ALE spaces. Recall that the theta-function θ_1 has the following well-known modular transformation properties [4]:

$$(13) \quad \theta_1(\tau, z + 1) = -\theta_1(\tau, z), \quad \theta_1(\tau, z + \tau) = -e^{-2\pi iz - \pi i \tau} \theta_1(\tau, z),$$

$$(14) \quad \theta_1(\tau + 1, z) = e^{\pi i/4} \theta_1(\tau, z), \quad \theta_1\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i\sqrt{\frac{\tau}{i}} e^{\frac{\pi i z^2}{\tau}} \theta_1(\tau, z).$$

From these it is straightforward to deduce the following transformation formulas:

$$\begin{aligned}
(15) \quad & Z_{X_n}(\tau, z+1; t_1, t_2) = (-1)^2 \cdot Z_{X_n}(\tau, z; t_1, t_2), \\
(16) \quad & Z_{X_n}(\tau, z+\tau; t_1, t_2) = e^{-\pi i(t_1+t_2)} \cdot (-e^{-2\pi i z - \pi i \tau})^2 \cdot Z_{X_n}(\tau, z; t_1, t_2), \\
(17) \quad & Z_{X_n}(\tau, z; t_1+1, t_2) = Z_{X_n}(\tau, z; \tau; t_1, t_2), \\
(18) \quad & Z_{X_n}(\tau, z; t_1+\tau, t_2) = e^{2\pi i z} \cdot Z_{X_n}(\tau, z; t_1, t_2), \\
(19) \quad & Z_{X_n}(\tau, z; t_1, t_2+1) = Z_{X_n}(\tau, z; t_1, t_2), \\
(20) \quad & Z_{X_n}(\tau, z; t_1, t_2+\tau) = e^{2\pi i z} \cdot Z_{X_n}(\tau, z; t_1, t_2), \\
(21) \quad & Z_{X_n}(\tau+1, z; t_1, t_2) = Z_{X_n}(\tau, z; t_1, t_2), \\
(22) \quad & Z_{X_n}\left(-\frac{1}{\tau}, \frac{z}{\tau}; \frac{t_1}{\tau}, \frac{t_2}{\tau}\right) = e^{2 \cdot \frac{\pi i}{\tau}(z^2 - z(t_1+t_2))} Z_{X_n}(\tau, z; t_1, t_2).
\end{aligned}$$

4. REGULARIZED ELLIPTIC GENERA OF X_r

We define regularized nonequivariant limit of equivariant elliptic genera of X_r by analyzing the singularity of Z_{X_r} . We are based on some results of Hoheneger-Iqbal [10] for which we present some new proofs.

4.1. Singularity of equivariant elliptic genera of X_r . Using the fact that

$$(23) \quad \theta_1(\tau, -z) = -\theta_1(\tau, z),$$

one can rewrite (10) in the following form:

$$(24) \quad Z_{X_r}(\tau, z; t_1, t_2) = \frac{\cdots}{\prod_{j=0}^r \theta_1(\tau, j t_1 + (j-r)t_2)}.$$

We first consider Z_{X_r} as a meromorphic function in t_1 . Since $\theta_1(\tau, z)$ has a first order zero at $z = 2\pi i(m\tau + n)$ for $m, n \in \mathbb{Z}$, so by (10), Z_{X_r} has possible first order poles when

$$(25) \quad j t_1 + (j-r)t_2 = m\tau + n,$$

i.e.,

$$(26) \quad t_1 = \frac{1}{j}(m\tau + n + (r-j)t_2).$$

for some $j = 1, \dots, r$, $m, n \in \mathbb{Z}$. However, by calculating the residues at these points, one finds that for many of them, the singularities of different terms in (10) cancel with each other, and only $t_1 = \frac{1}{r}(m\tau + n)$ ($m, n \in \mathbb{Z}$) are the first order poles of Z_{X_r} . The same holds for t_2 by

symmetry between t_1 and t_2 . So one can get an expansion of the form near $t_1 = t_2 = 0$:

$$(27) \quad Z_{X_r}(\tau, z; t_1, t_2) = \frac{1}{t_1 t_2} \sum_{m, n \geq 0} a_{m, n}(\tau, z) t_1^m t_2^n,$$

where $a_{m, n}(\tau, z)$ satisfy:

$$(28) \quad a_{m, n}(\tau, z) = a_{n, m}(\tau, z).$$

4.2. Equivariant elliptic genera restricted to a circle. The singularity of $Z_{X_r}(\tau, z; t_1, t_2)$ prevents us from taking the nonequivariant limit $\lim_{t_1 \rightarrow 0} \lim_{t_2 \rightarrow 0}$. To fix this problem, we take $t_1 = -t_2 = t$ to get:

$$(29) \quad Z_{X_r}(\tau, z; t, -t) = r \cdot \frac{\theta_1(\tau, z + rt)}{\theta_1(\tau, rnt)} \cdot \frac{\theta_1(\tau, z - rt)}{\theta_1(\tau, -rnt)},$$

or equivalently,

$$(30) \quad \begin{aligned} & Z_{X_r}(\tau, z; t, -t) \\ &= r y^{-1} \prod_{m=1}^{\infty} \frac{(1 - y e^{2\pi i r t} q^{m-1})(1 - y^{-1} e^{-2\pi i r t} q^m)}{(1 - e^{2\pi i r t} q^{m-1})(1 - e^{-2\pi i r t} q^m)} \\ & \quad \cdot \prod_{m=1}^{\infty} \frac{(1 - y e^{-2\pi i r t} q^{m-1})(1 - y^{-1} e^{2\pi i r t} q^m)}{(1 - e^{-2\pi i r t} q^{m-1})(1 - e^{2\pi i r t} q^m)}. \end{aligned}$$

This corresponds to restriction to the subtorus whose action preserves the holomorphic volume form (3). The expansion (27) now becomes an expansion of the following form:

$$(31) \quad Z_{X_r}(\tau, z; t, -t) = \sum_{g \geq 0} t^{2g-2} \alpha_{2g}(\tau, z).$$

The leading term is easily seen to be given by:

$$(32) \quad \alpha_0(\tau, z) = -\frac{1}{(2\pi)^2 r} \left(\frac{\theta_1(\tau, z)}{\eta(\tau)} \right)^2$$

Lemma 4.1. *The coefficients $\alpha_{2g}(\tau, z)$ are weak Jacobi forms of index 1 and weight $2g-2$. Furthermore, $b_{2g}(\tau, z) = \frac{\alpha_{2g}(\tau, z)}{\alpha_0(\tau, z)}$ are elliptic functions such that*

$$(33) \quad b_{2g}(\tau + 1, z) = b_{2g}(\tau, z),$$

$$(34) \quad b_{2g}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau^{2g} \cdot b_{2g}(\tau, z).$$

Proof. Easy consequence of the modular transformation properties of $Z_{X_r}(\tau, z; t_1, t_2)$ and the expansion (31). □

As noted in [10] in the case of \mathbb{C}^2 , $Z_{X_r}(\tau, z; t, -t)$ can be expressed in terms of Eisenstein series or Weierstrass \mathcal{P} -function:

Theorem 4.2. *The equivariant elliptic genus of X_r can be expressed as follows:*

$$(35) \quad Z_{X_r}(\tau, z; t, -t) = \frac{-r}{(2\pi i r t)^2} \cdot y^{-1} \left(\prod_{n=1}^{\infty} \frac{(1 - y q^{n-1})(1 - y^{-1} q^n)}{(1 - q^n)^2} \right)^2 \\ \cdot \exp \left(-2 \sum_{n=1}^{\infty} \frac{(rt)^{2n}}{(2n)!} \left(\frac{(2n-1)!}{z^{2n}} - \sum_{l=1}^{\infty} \frac{z^{2l}}{(2l)!} \frac{(2\pi i)^{2n+2l} B_{2n+2l}}{(2n+2l)} E_{2n+2l}(\tau) \right) \right),$$

where $E_{2k}(\tau)$ are the normalized Eisenstein series defined by:

$$(36) \quad E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

Alternatively,

$$(37) \quad Z_{X_r}(\tau, z; t, -t) = \frac{-r}{(2\pi r t)^2} \cdot \left(\frac{\theta_1(\tau, z)}{\eta(q)^3} \right)^2 \\ \cdot \exp \left(-2 \sum_{n=1}^{\infty} \frac{(rt)^{2n}}{(2n)!} \frac{\partial^{2n-2}}{\partial z^{2n-2}} \mathcal{P}(\tau, z) \right) \\ \cdot \exp \left(-2 \sum_{n=2}^{\infty} \frac{(2\pi i r t)^{2n}}{(2n)!} \frac{B_{2n}}{2n} E_{2n}(\tau) \right),$$

where $\mathcal{P}(\tau, z)$ is the Weierstrass \mathcal{P} -function given by:

$$(38) \quad \mathcal{P}(\tau, z) = \frac{1}{z^2} - \sum_{n=1}^{\infty} (2n+1) \frac{(2\pi i)^{2n+2} B_{2n+2}}{(2n+2)!} E_{2n+2}(\tau) z^{2n}.$$

Proof. In [10] a formula relating the Weierstrass \mathcal{P} -function and the Jacobi theta function was used. Here we show that these formulas can be derived by elementary series manipulations. We begin with:

$$\log(1 - e^v q^n) = - \sum_{m=1}^{\infty} \frac{q^{mn} e^{mv}}{m} = - \sum_{m=1}^{\infty} \frac{q^{mn}}{m} - \sum_{m=1}^{\infty} q^{mn} \sum_{k=1}^{\infty} m^{k-1} \frac{v^k}{k!} \\ = \log(1 - q^n) - \sum_{k=1}^{\infty} \frac{v^k}{k!} \sum_{m=1}^{\infty} m^{k-1} q^{mn}.$$

Note in passing that

$$(39) \quad \sum_{m=1}^{\infty} m^n x^n = \frac{\sum_{j=0}^{n-1} A(n, j) x^{j+1}}{(1-x)^{n+1}},$$

where $A(n, j)$ are Eulerian numbers and $\sum_{j=0}^{n-1} A(n, j) x^j$ are the Eulerian polynomials.

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\log(1 - e^v q^n) + \log(1 - e^{-v} q^n) \right) \\ &= 2 \sum_{n=1}^{\infty} \log(1 - q^n) - 2 \sum_{k=1}^{\infty} \frac{v^{2k}}{(2k)!} \sum_{m,n=1}^{\infty} m^{2k-1} q^{mn} \\ &= 2 \sum_{n=1}^{\infty} \log(1 - q^n) - 2 \sum_{k=1}^{\infty} \frac{v^{2k}}{(2k)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n. \end{aligned}$$

Now note

$$\begin{aligned} \frac{d}{dv} \log \frac{1 - e^v}{v} &= -\frac{e^v}{1 - e^v} - \frac{1}{v} = 1 + \frac{1}{v} \left(\frac{t}{e^v - 1} - 1 \right) \\ &= 1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} v^{k-1} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} v^{2k-1}. \end{aligned}$$

And so one can get:

$$(40) \quad \log(1 - e^v) = \log v + \pi i + \frac{1}{2}v + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} v^{2k}.$$

It follows that

$$\log(1 - e^v) + \log(1 - e^{-v}) = 2 \log v + 2 \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} v^{2k}.$$

Combining the above computations:

$$\begin{aligned}
& \log(1 - e^v) + \log(1 - e^{-v}) \\
& + 2 \sum_{n=1}^{\infty} \left(\log(1 - e^v q^n) + \log(1 - e^{-v} q^n) \right) \\
& = 2 \log v + 2 \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} v^{2k} \\
& + 4 \sum_{n=1}^{\infty} \log(1 - q^n) - 4 \sum_{k=1}^{\infty} \frac{v^{2k}}{(2k)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \\
& = 2 \log v + 4 \sum_{n=1}^{\infty} \log(1 - q^n) + 2 \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!(2n)} v^{2n} E_{2n}(\tau).
\end{aligned}$$

By (40) we get:

$$\log(1 - e^{u \pm v}) = \log(u \pm v) + \pi i + \frac{1}{2}(u \pm v) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} (u \pm v)^{2k},$$

and so

$$\begin{aligned}
& \log(1 - e^{u+v}) + \log(1 - e^{u-v}) \\
& = \log(u^2 - v^2) + 2\pi i + u + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} ((u+v)^{2k} + (u-v)^{2k}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left(\log(1 - e^{u+v} q^{n-1}) + \log(1 - e^{-u-v} q^n) \right. \\
& + \left. \log(1 - e^{u-v} q^{n-1}) + \log(1 - e^{-u+v} q^n) \right) \\
& = \log(u^2 - v^2) + 2\pi i + u + 4 \log(1 - q^n) \\
& + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} ((u+v)^{2k} + (u-v)^{2k}) E_{2k}(\tau) \\
& = 2 \log u + 2\pi i + u + 4 \log(1 - q^n) + 2 \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!(2k)} u^{2k} E_{2k}(\tau) \\
& - \sum_{n=1}^{\infty} \frac{1}{n} \frac{v^{2n}}{u^{2n}} + 2 \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \sum_{j=1}^k \frac{v^{2j}}{(2j)!} \frac{u^{2k-2j}}{(2k-2j)!} E_{2k}(\tau) \\
& = 2 \log \prod_{n=1}^{\infty} (1 - e^u q^{n-1}) (1 - e^{-u} q^n) \\
& - 2 \sum_{n=1}^{\infty} \frac{v^{2n}}{(2n)!} \left(\frac{(2n-1)!}{u^{2n}} - \sum_{l=0}^{\infty} \frac{u^{2l}}{(2l)!} \frac{B_{2n+2l}}{(2n+2l)} E_{2n+2l}(\tau) \right).
\end{aligned}$$

The $l = 0$ terms in the second line of the last equality are

$$2 \sum_{n=1}^{\infty} \frac{v^{2n}}{(2n)!} \frac{B_{2n}}{2n} E_{2n}(\tau).$$

After cancelling all these term and plugging in $u = 2\pi iz$ and $v = 2\pi irt$, one can prove (35). To prove (37), note

$$\begin{aligned}
& \frac{(2n-1)!}{u^{2n}} - \sum_{l=0}^{\infty} \frac{u^{2l}}{(2l)!} \frac{B_{2n+2l}}{(2n+2l)} E_{2n+2l}(\tau) \\
& = \frac{\partial^{2n-2}}{\partial z^{2n-2}} \mathcal{P}(\tau, z) - \delta_{n,1} \frac{B_2}{2} E_2(\tau).
\end{aligned}$$

□

Let

$$(41) \quad G_{2k}(\tau) = -\frac{(2\pi i)^{2k} B_{2k}}{(2k)!} E_{2k}.$$

Then (37) can be rewritten as follows:

$$\begin{aligned}
 Z_{X_r}(\tau, z; t, -t) &= \frac{r}{(2\pi r t)^2} \cdot \left(\frac{\theta_1(\tau, z)}{\eta(q)^3} \right)^2 \\
 &\cdot \exp \left(-2 \sum_{n=1}^{\infty} \frac{(rt)^{2n}}{(2n)!} \frac{\partial^{2n-2}}{\partial z^{2n-2}} \mathcal{P}(\tau, z) \right) \\
 &\cdot \exp \left(\sum_{n=2}^{\infty} \frac{(rt)^{2n}}{n} G_{2n}(\tau) \right).
 \end{aligned}
 \tag{42}$$

To find the coefficients $\alpha_{2g}(\tau, z)$ in (31), we use the following result due to [10]:

$$\begin{aligned}
 &\exp \left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \frac{\partial^{2n-2}}{\partial z^{2n-2}} \mathcal{P}(\tau, z) \right) \cdot \exp \left(\sum_{n=2}^{\infty} \frac{t^{2n}}{n} G_{2n}(\tau) \right) \\
 &= 1 - \mathcal{P}(\tau, z)t^2 + \sum_{n \geq 2} (2n-1)G_{2n}(\tau)t^{2n}.
 \end{aligned}
 \tag{43}$$

We will present two new proofs here. Let us first rewrite this formula. Since

$$\mathcal{P}(\tau, z) = \frac{1}{z^2} + \sum_{n \geq 2} (2n-1)G_{2n}(\tau)z^{2n-2},
 \tag{44}$$

the above formula can be rewritten as

$$\begin{aligned}
 &\mathcal{P}(\tau, t) - \mathcal{P}(\tau, z) \\
 &= \frac{1}{t^2} \exp \left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \frac{\partial^{2n-2}}{\partial z^{2n-2}} \mathcal{P}(\tau, z) + \sum_{n=2}^{\infty} \frac{t^{2n}}{n} G_{2n}(\tau) \right).
 \end{aligned}
 \tag{45}$$

The right-hand side of this formula can be written as:

$$\begin{aligned}
 &\frac{1}{t^2} \exp \left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \left(\frac{(2n-1)!}{z^{2n}} + \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l} \right) \right) \\
 &= \left(\frac{1}{t^2} - \frac{1}{z^2} \right) \cdot \exp \left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l} \right),
 \end{aligned}$$

by explicitly expanding the second term on the right-hand side, one can see that it can be written in the following form:

$$\frac{1}{t^2} - \mathcal{P}(\tau, z) + \sum_{n \geq 2} \beta_{2n}(\tau, z)t^{2n-2},
 \tag{46}$$

where $\beta_{2n}(\tau, z)$ is holomorphic in both τ and z .

Lemma 4.3. *For each $n \geq 2$, $\beta_{2n}(\tau, z)$ is independent of z and is a modular form of weight $2n$ for $SL_2(\mathbb{Z})$, and so it can be written simply as $\beta_{2n}(\tau)$.*

Proof. Similar to Lemma 4.1, one can show that $\beta_{2n}(\tau, z)$ is an elliptic function in z :

$$\beta_{2n}(\tau, z+1) = \beta_{2n}(\tau, z+\tau) = \beta_{2n}(\tau, z),$$

and

$$\beta_{2n}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau^{2n} \beta_{2n}(\tau, z).$$

But β_{2n} is holomorphic in z , so β_{2n} is independent of z , and therefore it is a modular form of weight $2n$. \square

Now we have:

$$\begin{aligned} & \frac{1}{t^2} - \mathcal{P}(\tau, z) + \sum_{n \geq 2} \beta_{2n}(\tau) t^{2n-2} \\ (47) \quad &= \left(\frac{1}{t^2} - \frac{1}{z^2}\right) \cdot \exp\left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l}\right). \end{aligned}$$

Now we take $\frac{\partial}{\partial t}$ on both sides:

$$\begin{aligned} & -\frac{2}{t^3} + \sum_{n \geq 2} (2n-2) \beta_{2n}(\tau) t^{2n-3} \\ &= -\frac{2}{t^3} \cdot \exp\left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l}\right) \\ &+ \left(\frac{1}{t^2} - \frac{1}{z^2}\right) \cdot \exp\left(-2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l}\right) \\ &\cdot \left(-2 \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l}\right) \\ &= \left(\frac{1}{t^2} - \mathcal{P}(\tau, z) + \sum_{n \geq 2} \beta_{2n}(\tau) t^{2n-2}\right) \\ &\cdot \left(-\frac{2}{t} \sum_{n=0}^{\infty} \frac{t^{2n}}{z^{2n}} - 2 \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \sum_{l=1}^{\infty} \frac{(2n+2l-1)!}{(2l)!} G_{2n+2l}(\tau) z^{2l}\right). \end{aligned}$$

Since the left-hand side is independent of the variable z , so after taking the terms on the right-hand side of the second equality independent of

z , we get

$$\begin{aligned}
& -\frac{2}{t^3} + \sum_{n \geq 2} (2n-2) \beta_{2n}(\tau) t^{2n-3} \\
&= -\frac{2}{t^3} + 2 \sum_{n \geq 2} (2n-1) G_{2n}(\tau) t^{2n-3} + 2 \sum_{n \geq 1} \frac{(2n+1)!}{(2n-1)!2!} G_{2n+2} t^{2n-1} \\
&- 2 \sum_{n \geq 2} \beta_{2n}(\tau) t^{2n-3} \\
&= -\frac{2}{t^3} + \sum_{n \geq 2} (2n-2)(2n-1) G_{2n}(\tau) t^{2n-3}.
\end{aligned}$$

So we get:

$$(48) \quad \beta_{2n}(\tau) = (2n-1) G_{2n}(\tau).$$

This completes our first proof.

For the second proof, rewrite (43) in the following form:

$$(49) \quad \mathcal{P}(\tau, t) - \mathcal{P}(\tau, z) = (2\pi)^2 \left(\frac{\theta_1(\tau, z)}{\eta(q)^3} \right)^{-2} \frac{\theta_1(\tau, z+t)}{\theta_1(\tau, t)} \cdot \frac{\theta_1(\tau, z-t)}{\theta_1(\tau, -t)}.$$

This can be proved using the following formula in terms of Weierstrass σ -function [4, p. 55]:

$$(50) \quad \mathcal{P}(\tau, u) - \mathcal{P}(\tau, v) = -\frac{\sigma(\tau, u+v)\sigma(\tau, u-v)}{\sigma^2(\tau, u)\sigma^2(\tau, v)},$$

and the following formula of σ in terms of theta function [4, p. 60]:

$$(51) \quad \sigma(\tau, u) = \frac{\theta_1(\tau, z)}{\partial_z \theta_1(\tau, 0)} e^{\eta z^2}.$$

As a consequence of (43), one gets:

Theorem 4.4. *The expansion of $Z_{X_r}(\tau, z; t, -t)$ is given by:*

$$\begin{aligned}
(52) \quad Z_{X_r}(\tau, z; t, -t) &= \frac{-1}{(2\pi t)^2 r} \left(\frac{\theta_1(\tau, z)}{\eta(\tau)} \right)^2 \\
&\cdot \left(1 - \mathcal{P}(\tau, z)(rt)^2 + \sum_{n \geq 2} (2n-1) G_{2n}(\tau) (rt)^{2n} \right).
\end{aligned}$$

4.3. Regularized elliptic genera of X_r . We define the regularized elliptic genus of X_r to be

$$(53) \quad Z_{X_r}^{reg}(\tau, z) := \lim_{t \rightarrow 0} \left(Z_{X_r}(\tau, z; t, -t) - \frac{-1}{(2\pi t)^2 r} \left(\frac{\theta_1(\tau, z)}{\eta(\tau)} \right)^2 \right).$$

I.e.,

$$(54) \quad Z_{X_r}^{reg}(\tau, z) = \frac{r}{(2\pi)^2} \left(\frac{\theta_1(\tau, z)}{\eta(\tau)} \right)^2 \mathcal{P}(\tau, z).$$

It is a weak Jacobi form of weight 0 and index 2, so it is a multiple of $Z_{K3}(\tau, z)$, the elliptic genus of a K3 surface [5, 3].

To conclude, we remark that it should be interesting to consider

$$(55) \quad Z_{X_r}(\tau, z; t\sqrt{\beta}, -\frac{t}{\sqrt{\beta}}) = -\frac{1}{t^2} \sum_{m,n \geq 0} a_{m,n}(\tau, z) (-1)^n t^{m+n} \sqrt{\beta}^{m-n},$$

and resum the right-hand side as follows:

$$(56) \quad \sum_{m+n=k} a_{m,n}(\tau, z) (-1)^n \sqrt{\beta}^{m-n} = \sum_{j=0}^k c_j(\tau, z) (\sqrt{\beta} - \frac{1}{\sqrt{\beta}})^j.$$

It is also interesting to consider the equivariant elliptic genera of toric Calabi-Yau manifolds of dimensions > 3 . We hope to address such problems in the future.

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